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On Characterizations of Proper Efficiency for Nonconvex Multiobjective Optimization

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Abstract. In this paper, nonconvex multiobjective optimization problems are studied. New characterizations of a properly efficient solution in the sense of Geoffrion's are established in terms of the stability of one scalar optimization problem and the existence of an exact penalty function of a scalar constrained program, respectively. One of the characterizations is applied to derive necessary conditions for a properly efficient control-parameter pair of a nonconvex multiobjective discrete optimal control problem with linear constraints.

Key words: Multiobjective optimization, Properly efficient solution, Stability, Multicriteria discrete time optimal control

1. Introduction

Consider the following multiobjective programming problem:

$$\text{(MOP)} \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X, \end{array}$$

where $X \subset R^n$ is a nonempty closed set, $f = (f_1, \dots, f_l) : X \to R^l$ is a vector-valued function. The objective space R^l is ordered by the nonnegative orthant R^l_+, that is, $\forall y^1, y^2 \in R^l$, $y^1 \leqslant_{R^l_+} y^2$ if and only if $y^2 - y^1 \in R^l_+$.

We quote the following basic concepts in vector optimization from Sawaragi et al., [16].

DEFINITION 1.1. A point $x^* \in X$ is called an efficient solution of (MOP) if there exists no $x \in X$ such that

$$f(x) - f(x^*) \in -R^l_+ \backslash \{0\}.$$

The set of all the efficient solutions of (MOP) is denoted by E.

DEFINITION 1.2. A point $x^* \in X$ is called a properly efficient solution of (MOP) (in the sense of Geoffrion's) if $x^* \in E$ and there exists a real number $M > 0$ such

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that for any $x \in X$ if $f_i(x) < f_i(x^*)$ for some $i \in \{1, \ldots, l\}$, there exists a $j \in \{1, \ldots, l\} \backslash \{i\}$ such that

$$0 < \frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leqslant M.$$

The set of all the properly efficient solutions of (MOP) is denoted by PE.

REMARK 1.1. It is easy to verify that $x^* \in$ PE if and only if $x^* \in$ E and $\exists M > 0$ such that for any $x \in X$ with $f_i(x) < f_i(x^*)$, for some $i \in \{1, \ldots, l\}$, one has

$$0 < \frac{f_i(x^*) - f_i(x)}{\max\limits_{1 \leqslant j \leqslant l} \{f_j(x) - f_j(x^*)\}} \leqslant M.$$

Let $x^* \in E$. Benson and Morin [1] provided a characterization when x^* is a properly efficient solution in terms of the stability of the following scalar programs:

$$\begin{array}{lll} (P_i) & \min & f_i(x) \\ & \text{s.t.} & x \in X, \\ & & f_j(x) - f_j(x^*) \leqslant 0, \forall j \in \{1, \ldots, l\} \backslash \{i\}, \end{array}$$

where $i = 1, \ldots, l$.

The conclusion in Benson and Morin [1] is based on the assumption that X is convex and each f_i is convex on X. On the other hand, Choo and Atkins [3] characterized Geoffrion's proper efficiency as the best approximation to the ideal point without any convexity assumption. A general notion of proper efficiency is defined in Borwein [2] in a locally convex space ordered by a closed convex cone, which is weaker than Geoffrion's version even if the locally convex space reduces to a finite dimensional space R^l and the ordering cone is R^l_+. Characterizations of Borwein's proper efficiency were provided in Borwein [2] in terms of scalarization under the assumptions of convexity. Jahn [8] further characterized Borwein's proper efficiency as minimal solutions of appropriate approximation problems without any convexity assumptions in normed spaces.

The study of multiobjective optimal control problems started in Zadeh [21]. Since then, necessary and sufficient conditions as well as various methods were developed for multiobjective optimal control, see, e.g., Li [10], Li [11], Liao and Li [12], Liu [13], Salukvadze [15], Toivonen [18], Yang and Teo [19], Yu and Leitmann [20]. It is known in multiobjective optimization that properly efficient solutions are much 'better' than general efficient solutions. Moreover, generally speaking, the set of efficient solutions is too big and hard to obtain while the set of properly efficient solutions is much smaller and so easier to be found. Zalmai [22] studied necessary and sufficient conditions for a properly efficient control to a multiobjective continuous time fractional control problem by parameterization and linear scalarization under the condition of convexity. However, to the best knowledge of us, no work on properly efficient solutions to a general constrained multiobjective optimal control problem has been conducted in the literature.

In this paper, we study properties of properly efficient solutions (in the sense of Geoffrion's) of (MOP) without any convexity requirement. The first characterization of proper efficiency is obtained via exact penalization of a scalar constrained program, which is closely related to (MOP), while the second characterization is established via the stability of a related scalar optimization problem. In order to obtain these characterizations of proper efficiency, we extend the characterization of efficient solutions given in Deng [5]. As a bridge, the equivalence of the stability of a scalar constrained program with the existence of an exact penalty function is also established. These results are obtained by using a penalty function, while the characterizations in Choo and Atkins [3], Borwein [2] and Jahn [8] employed a set of related scalar optimization problems (P_i).

Finally, we apply the characterization of the proper efficiency for (MOP) in terms of the existence of an exact penalty function to derive necessary conditions for a properly efficient control-parameter pair to a multicriteria discrete time optimal control problem with linear state equations and linear constraints.

2. Stability of scalar constrained optimization

Consider the following scalar constrained optimization problem:

$$\begin{array}{lll} (P') & \min & g(x) \\ & \text{s.t.} & x \in X, \\ & & h_j(x) \leqslant 0,\ j = 1, \dots, m, \end{array}$$

where $X \subset R^n$ is a nonempty closed set, $g, h_j (j = 1, \dots, m) : X \to R^1$ are real-valued functions.

Associate (P') with the following perturbed optimization problem:

$$\begin{array}{lll} (P'_u) & \min & g(x) \\ & \text{s.t.} & x \in X, \\ & & h_j(x) \leqslant u_j,\ j = 1, \dots, m, \end{array}$$

where $u = (u_1, \dots, u_m) \in R^m_+$ with norm $\|u\| = \sum_{j=1}^m |u_j|$ in R^m.

Let $\gamma(u)$ denote the optimal value of (P'_u). Then $\gamma(0)$ is the optimal value of (P').

The following definition of stability of a scalar constrained optimization problem is equivalent to the usual definition of stability (see, e.g., Sawaragi et al. [16], Benson and Morin [1] and Rosenberg [14]) though they are different in form.

DEFINITION 2.1. (P') is said to be stable if $\gamma(0)$ is finite and there exists a real number $M > 0$ such that

$$\frac{\gamma(u) - \gamma(0)}{\|u\|} \geqslant -M, \forall u \in R^m_+.$$

Let $x^* \in X$ be an optimal solution of (P'). Consider the following penalty function of (P'):

$$p(x) = g(x) + r\max\{h_1(x), \dots, h_m(x), 0\}, x \in X, r > 0.$$

$p(x)$ is said to be an exact penalty function for (P') if x^* minimizes $p(x)$ on X for a finite $r > 0$.

LEMMA 2.1. *Consider (P') and (P'_u). Let $x^* \in X$ be an optimal solution of (P'). Then there exists $r^* > 0$ such that the penalty function*

$$p(x) = g(x) + r^* max\{h_1(x), \dots, h_m(x), 0\}$$

is exact if and only if (P') is stable.

Proof. 'Sufficiency.' Let (P') be stable. Suppose to the contrary that there exists $r_k \to +\infty$ and $x_k \in X$ such that

$$g(x_k) + r_k \max\{h_1(x_k), \dots, h_m(x_k), 0\} < g(x^*), \forall k. \tag{1}$$

Since

$$\sum_{i=1}^{m} h_j^+(x_k) \leqslant m\max\{h_1(x_k), \dots, h_m(x_k), 0\}, \forall k,$$

where $h_j^+(x_k) = \max\{h_j(x_k), 0\}$, $j = 1, \dots, m$, we deduce from (1) that

$$g(x_k) + r_k/m \sum_{j=1}^{m} h_j^+(x_k) < g(x^*), \forall k. \tag{2}$$

By (2), we get

$$\sum_{j=1}^{m} h_j^+(x_k) > 0.$$

Otherwise, x_k is a feasible solution of (P') and

$$g(x_k) < g(x^*),$$

which contradicts the fact that x^* is an optimal solution of (P'). Let

$$u_j^k = h_j^+(x_k), j = 1, \dots, m, \forall k, u^k = (u_1^k, \dots, u_m^k). \tag{3}$$

Then

$$u^k \in R_+^m, \|u^k\| > 0, h_j(x_k) \leqslant u_j^k, j = 1, \dots, m,$$

that is, x_k is a feasible solution of (P_{u^k}). It follows from (2) and (3) that

$$\frac{\gamma(u^k)-\gamma(0)}{\|u^k\|} \leqslant \frac{g(x_k)-g(x^*)}{\|u^k\|} \leqslant -\frac{r_k}{m},$$

which contradicts the stability of (P') because $r_k \to +\infty$ as $k \to +\infty$. This proves the sufficiency.

'Necessity.' Suppose to the contrary that (P') is not stable. Then $\exists u^k = (u_1^k, \ldots, u_m^k) \in R_+^m$, $\|u^k\| > 0$ with $\|u^k\| \to 0$ such that

$$\frac{\gamma(u^k)-\gamma(0)}{\|u^k\|} \to -\infty.$$

It follows that $\exists x_k \in X$ satisfying

$$h_j(x_k) \leqslant u_j^k,\ j = 1, \ldots, m \tag{4}$$

such that

$$\frac{g(x_k)-g(x^*)}{\|u^k\|} \to -\infty. \tag{5}$$

On the other hand, $\exists r^* > 0$ such that x^* minimizes $p(x) = g(x) + r^*\max\{h_1(x), \ldots, h_m(x), 0\}$ on X, namely,

$$g(x^*) \leqslant g(x) + r^*\max\{h_1(x), \ldots, h_m(x), 0\}, \forall x \in X.$$

Therefore,

$$\begin{aligned} g(x^*) &\leqslant g(x_k) + \max\{h_1(x_k), \ldots, h_m(x_k), 0\} \\ &\leqslant g(x_k) + r^* \sum_{j=1}^{m} h_j^+(x_k), \forall k. \\ &\leqslant g(x_k) + r^* \sum_{j=1}^{m} u_j^k \\ &= g(x_k) + r^*\|u^k\|, \end{aligned}$$

which implies

$$\frac{g(x_k)-g(x^*)}{\|u^k\|} \geqslant -r^*,$$

a contradiction to (5). The proof is complete. □

3. Characterizations of proper efficiency

Let $\lambda = (\lambda_1, \ldots, \lambda_l) \in \text{int}R^l_+$, where $\text{int}R^l_+$ denotes the interior of R^l_+. Consider the following constrained scalar optimization problem:

$$\begin{aligned} (P_\lambda) \quad \min \quad & \sum_{i=1}^{l} \lambda_i f_i(x) \\ \text{s.t.} \quad & x \in X, \\ & f_i(x) \leqslant f_i(x^*), i = 1, \ldots, l. \end{aligned}$$

LEMMA 3.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_l) \in intR^l_+$. Then $x^* \in E$ if and only if x^* solves (P_λ).*

Proof. Let $x^* \in$ E. It is obvious that x^* is a feasible solution of (P_λ). Suppose to the contrary that there exists $\overline{x} \in X$, which is a feasible solution of (P_λ), that is,

$$f_i(\overline{x}) \leqslant f_i(x^*), i = 1, \ldots, l. \tag{6}$$

such that

$$\sum_{i=1}^{l} \lambda_i f_i(\overline{x}) < \sum_{i=1}^{l} \lambda_i f_i(x^*).$$

Then there exists $i^* \in \{1, \ldots, l\}$ such that

$$f_{i^*}(\overline{x}) < f_{i^*}(x^*). \tag{7}$$

The combination of (6) and (7) contradicts the fact that $x^* \in$E. This proves the necessity. Now we prove the sufficiency by contradiction. Let x^* solve (P_λ). Suppose that there exists $\overline{x} \in X$ such that

$$f_i(\overline{x}) \leqslant f_i(x^*), i = 1, \ldots, l.$$

with at least one strict inequality. This implies that $\overline{x}$ is a feasible solution of (P_λ). Since $\lambda = (\lambda_1, \ldots, \lambda_l) \in \text{int}R^l_+$, it follows that

$$\sum_{i=1}^{l} \lambda_i f_i(\overline{x}) < \sum_{i=1}^{l} \lambda_i f_i(x^*).$$

Hence, x^* is not an optimal solution of (P_λ). The proof is complete. □

REMARK 3.1. This lemma extends Deng's characterization of efficient solutions (see Theorem 2.1, (a)⇔(b) in Deng [5]) where $\lambda = (1, \ldots, 1) \in \text{int}R^l_+$.

THEOREM 3.1. *Let $\lambda \in intR^l_+$. Consider (MOP) and (P_λ). Let $x^* \in E$. Then $x^* \in$ PE if and only if $\exists r^*_\lambda > 0$ such that x^* minimizes*

$$p_\lambda(x) = \sum_{i=1}^{l} \lambda_i f_i(x) + r^*_\lambda max\{f_1(x) - f_1(x^*), \ldots, f_l(x) - f_l(x^*), 0\}$$

on X, namely, the penalty function p_λ is exact for (P_λ).

Proof. ‘Sufficiency.’ Suppose that $\overline{x} \in X$ and $i^* \in \{1, \dots, m\}$ is such that

$$f_{i^*}(\overline{x}) < f_{i^*}(x^*). \tag{8}$$

Since x^* minimizes $p_\lambda(x)$ on X, that is,

$$\sum_{i=1}^{l} \lambda_i f_i(x^*) \leqslant \sum_{i=1}^{l} \lambda_i f_i(x) + r_\lambda^* \max\{f_1(x) - f_1(x^*), \dots, f_l(x) - f_l(x^*), 0\}, \forall x \in X,$$

we have

$$\begin{aligned} &\lambda_{i^*}[f_{i^*}(x^*) - f_{i^*}(\overline{x})] \\ &\leqslant \left(\sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \lambda_i \right) \max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*), 0\} \\ &\quad + r_\lambda^* \max\{f_1(x) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*), 0\} \\ &= \left(r_\lambda^* + \sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \lambda_i \right) \max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*), 0\}. \end{aligned} \tag{9}$$

Noting that $x^* \in$ E, it follows from (8) that

$$\max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*), 0\} > 0. \tag{10}$$

Hence,

$$\begin{aligned} &\max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*), 0\} \\ &\quad = \max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*)\} \end{aligned} \tag{11}$$

Combining (9) with (19) and (11) yields

$$0 < \frac{f_{i^*}(x^*) - f_{i^*}(\overline{x})}{\max\{f_1(\overline{x}) - f_1(x^*), \dots, f_l(\overline{x}) - f_l(x^*)\}} \leqslant \frac{r_\lambda^* + \sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \lambda_i}{\lambda_{i^*}}.$$

By Remark 1.1, $x^* \in$ PE.

‘Necessity.’ Let $x^* \in$PE. By Lemma 3.1, x^* solves (P_λ). Suppose to the contrary that there exist $r_k \to +\infty$ and $x_k \in X$ such that

$$\sum_{i=1}^{l} \lambda_i f_i(x_k) + r_k \max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\} < \sum_{i=1}^{l} \lambda_i f_i(x^*), \forall k. \tag{12}$$

Let $I_k = \{i \in \{1, \dots, l\} : f_i(x_k) < f_i(x^*)\}$ and $\bar{I}_k = \{1, \dots, l\} \backslash I_k$. Then by (12) and $x^* \in E$, we see that $I_k \neq \emptyset$ and $\bar{I}_k \neq \emptyset$. In addition,

$$\max\{f_1(x_k) - f_1(x^*), ..., f_l(x_k) - f_l(x^*), 0\} = \max_{i \in \bar{I}_k}\{f_i(x_k) - f_i(x^*)\} > 0$$

since $x^* \in E$. It follows from (12) that

$$\sum_{i \in I_k} \lambda_i(f_i(x^*) - f_i(x_k)) \tag{13}$$

$$> \sum_{i \in \bar{I}_k} \lambda_i(f_i(x_k) - f_i(x^*)) + r_k \max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\} \tag{14}$$

$$\geqslant r_k \max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\}. \tag{15}$$

Thus,

$$\frac{\sum_{i \in I_k} \lambda_i(f_i(x^*) - f_i(x_k))}{\max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\}} > r_k. \tag{16}$$

As $x^* \in$ PE, we deduce that there exists a real number $M > 0$ such that

$$\begin{aligned} &\frac{f_i(x^*) - f_i(x_k)}{\max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*)\}} \\ &\quad = \frac{f_i(x^*) - f_i(x_k)}{\max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\}} \\ &\quad \leqslant M, \forall i \in I_k. \end{aligned}$$

As a result,

$$\frac{\sum_{i \in I_k} \lambda_i(f_i(x^*) - f_i(x_k))}{\max\{f_1(x_k) - f_1(x^*), \dots, f_l(x_k) - f_l(x^*), 0\}} \leqslant \sum_{i=1}^{l} \lambda_i M. \tag{17}$$

Inequality (17) contradicts (16) because $r_k \to +\infty$ as $k \to +\infty$. The proof is complete. □

REMARK 3.2. *The result of Theorem 3.1 can be seen as a characterization of proper efficiency in terms of exact penalization for* (P_λ).

THEOREM 3.2. *Let* $\lambda \in intR^l_+$. *Consider (MOP) and* (P_λ). *Let* $x^* \in E$. *Then* $x^* \in$ *PE if and only if* (P_λ) *is stable.*

Proof. By Lemma 3.1, x^* solves (P_λ). By Theorem 3.1, $x^* \in$ PE if and only if x^* minimizes p_λ on X. Further, by Lemma 2.1, x^* minimizes p_λ if and only if (P_λ) is stable. So $x^* \in$ PE if and only if (P_λ) is stable. The proof is complete. □

4. Necessary conditions for a properly efficient control-parameter pair to a multicriteria discrete time optimal control problem

In this section, we apply Theorem 3.1 to derive necessary conditions for a properly efficient control-parameter pair to a multicriteria discrete time optimal control problem with linear state equations and linear constraints.

Consider the following multicriteria discrete time optimal control problem:

$$\text{(MDOC)}\ \min f(x(\cdot), u(\cdot), z) = (f_1(x(\cdot), u(\cdot), z), \dots, f_l(x(\cdot), u(\cdot), z))$$

subject to the linear state difference equation

$$x(k+1) = Ax(k) + Bu(k) + Cz, k = 0, 1, ..., N-1, x(0) = x^0(z) \quad (18)$$

and the following linear constraints:

$$a_i^T u(\cdot) + b_i^T z = c_i, i \in E, \quad (19)$$

$$a_j^T u(\cdot) + b_j^T z \leqslant c_j, j \in I, \quad (20)$$

where the objective space R^l is still ordered by R^l_+, $x = (x_1, \dots, x_n)^T : \{1, \dots, N\} \to R^n$, $u = (u_1, \dots, u_r)^T : \{0, \dots, N-1\} \to R^r$ and $z = (z_1, \dots, z_s)^T \in R^s$ are state, control and system parameter vectors respectively, $x^0 = (x_1^0, \dots, x_n^0)^T : R^s \to R^n$, and $A_{n\times n}, B_{n\times r}, C_{n\times s}$ are matrices, $a_i, a_j \in R^{Nr}$, $b_i, b_j \in R^s$, $c_i, c_j \in R$, E and I are finite sets such that $I \bigcap E = \emptyset$, $f_i : R^{(n+r)N+s} \to R$ are real functions, $i = 1, \dots, l$, see Teo et al. [17].

REMARK 4.1. When $l = 1$, the corresponding continuous time optimal control model was discussed in Teo et al. [17] (p.149). As noted in [17], this model includes several important optimal control and optimal parameter selection models as special cases. A software package for solving their general single objective (continuous time and discrete time) optimal control problems can be found in Jennings et al. [9]. In Yang and Teo [19], a bicriteria ($l = 2$) discrete time optimal control problem was studied.

A control-parameter pair $(u(\cdot), z)$ is said to be feasible if (18)–(20) are satisfied. Let $\mathcal{F}$ be the set of feasible control-parameter pairs $(u(\cdot), z)$.

DEFINITION 4.1. A feasible control-parameter pair $(u^*(\cdot), z^*)$ of (MDOC) is said to be efficient if there exists no feasible control-parameter $(u(\cdot), z)$ such that

$$f(x(\cdot), u(\cdot), z) - f(x^*(\cdot), u^*(\cdot), z^*) \in -R^l_+ \backslash \{0\},$$

where $x^*(\cdot)$, $x(\cdot)$ are corresponding (unique) solution of (18) with respect to $(u^*(\cdot), z^*)$ and $(u(\cdot), z)$, respectively.

DEFINITION 4.2. A feasible control-parameter pair $(u^*(\cdot), z^*)$ of (MDOC) is said to be properly efficient if it is efficient and there exists a real number $M > 0$ such that for any $i \in \{1, \dots, l\}$ and $(u(\cdot), z) \in \mathcal{F}$ with $f_i(x(\cdot), u(\cdot), z) < f_i(x^*(\cdot), u^*(\cdot), z^*)$, there exists $j \in \{1, \dots, l\}\backslash\{i\}$ satisfying

$$0 < \frac{f_i(x^*(\cdot), u^*(\cdot), z^*) - f_i(x(\cdot), u(\cdot), z)}{f_j(x(\cdot), u(\cdot), z) - f_j(x^*(\cdot), u^*(\cdot), z^*)} \leqslant M.$$

DEFINITION 4.3. A feasible control-parameter pair $(u^*(\cdot), z^*)$ of (MDOC) is said to be locally efficient if there exists a neighbourhood V of $(u^*(\cdot), z^*)$ such that there exists no control-parameter $(u(\cdot), z) \in V \bigcap \mathcal{F}$ such that

$$f(x(\cdot), u(\cdot), z) - f(x^*(\cdot), u^*(\cdot), z^*) \in -R^l_+\backslash\{0\}.$$

DEFINITION 4.4. A feasible control-parameter pair $(u^*(\cdot), z^*)$ of (MDOC) is said to be locally properly efficient if there exists a neighbourhood V of $(u^*(\cdot), z^*)$ such that it is efficient to (MDOC) on $V \bigcap \mathcal{F}$ and there exists a real number $M > 0$ such that for any $i \in \{1, \dots, l\}$ and $(u(\cdot), z) \in V \bigcap \mathcal{F}$ with $f_i(x(\cdot), u(\cdot), z) < f_i(x^*(\cdot), u^*(\cdot), z^*)$, there exists $j \in \{1, \dots, l\}\backslash\{i\}$ satisfying

$$0 < \frac{f_i(x^*(\cdot), u^*(\cdot), z^*) - f_i(x(\cdot), u(\cdot), z)}{f_j(x(\cdot), u(\cdot), z) - f_j(x^*(\cdot), u^*(\cdot), z^*)} \leqslant M.$$

In the following we shall use some concepts in nonsmooth analysis for locally Lipschitz real-valued functions, and we refer readers to Clarke [4] for details.

Let $\varphi : R^n \to R$ be a locally Lipschitz real-valued function. We denote by $\partial^0\varphi(x)$ the Clarke generalized subdifferential of φ at $x \in R^n$.

To obtain necessary conditions for a properly efficient control-parameter to (MDOC), as in Yang and Teo [19], we first consider optimality conditions for a general scalar discrete time optimal control problem:

$$\begin{array}{lll} \text{(DOC)} & \min & \phi(x(\cdot), u(\cdot), z) \\ & \text{s.t.} & (u(\cdot), z) \in \mathcal{F}, \end{array}$$

where $\phi : R^{(n+r)N+s} \to R$ is locally Lipschitz.

Since $x(\cdot)$ is uniquely determined by $(u(\cdot), z)$, ϕ can be seen as a function of $(u(\cdot), z)$, namely, ϕ can be seen as a function of $(u(\cdot), z)$ from R^{rN+s} to R. We quote some results from Yang and Teo [19].

LEMMA 4.1 (Yang and Teo [19], Theorem 3.1). *Let $\phi : R^{rN+s} \to R$ be subdifferentially regular at $(u(\cdot), z)$. Then the subdifferential of ϕ at $(u(\cdot), z)$ is approximated by*

$$\partial^0\phi(x(\cdot,u(\cdot),z),u(\cdot),z)\subseteq\{(g_0,\ldots,g_{N-1},h_0):$$
$$g_k=\sum_{t=k+1}^{N}A^{t-k-1}Bg_{x(t)}+g_{u(k)},\,k=0,\ldots,N-1,$$
$$h_0=\sum_{k=1}^{N}\left[A^k\frac{\partial x^0(z)}{\partial z}+(A^{k-1}+A^{k-2}+\cdots+I)C\right]g_{x(k)}+g_z,$$
$$(g_{x(1)},\ldots,g_{x(N)},g_{u(0)},\ldots,g_{u(N-1)},g_z)\in\partial^0\phi(x(\cdot),u(\cdot),z)\},$$

where $x(\cdot,u(\cdot),z)$ *denotes the solution of (19) and (20) with respect to* $(u(\cdot),z)$.
Define

$$CE(u(\cdot),z)=\{(\delta u(\cdot),\delta z)\in R^{rN+s}:a_i^T\delta u(\cdot)+b_i^T\delta z=0,i\in I\},$$

$$CI(u(\cdot),z)=\{(\delta u(\cdot),\delta z)\in R^{rN+s}:a_j^T\delta u(\cdot)+b_j^T\delta z\leqslant 0,\,j\in I(u(\cdot),z)\},$$

where $I(u(\cdot),z)=\{j\in I:a_j^Tu(\cdot)+b_j^Tz=c_j\}$.

LEMMA 4.2 (Yang and Teo [19], Theorem 3.2). *Assume that* $\phi(x(\cdot,u(\cdot),z),u(\cdot),z)$ *is subdifferentially regular at* $(u^*(\cdot),z^*)$. *If* $(u^*(\cdot),z)$ *is optimal of the scalar (DOC) problem, then*

$$\inf_{(\delta u(\cdot),\delta z)}\max_{\tilde g}\left\{\sum_{k=0}^{N-1}\left(\sum_{t=k+1}^{N}A^{t-k-1}Bg_{x(t)}+g_{u(k)}\right)^T\delta u(\cdot)+\right.$$
$$\left.\sum_{k=1}^{N}\left(\left[A^k\frac{\partial x^0(z)}{\partial z}+(A^{k-1}+A^{k-2}+\cdots I)C\right]g_{x(k)}+g_z\right)^T\delta z\right\}\geqslant 0,$$

where

$$(\delta u(\cdot),\delta z)\in CI(u^*(\cdot),z^*)\bigcap CE(u^*(\cdot),z^*),$$

$$\tilde g=(g_{x(1)},\ldots,g_{x(N)},g_{u(0)},\ldots,g_{u(N-1)},g_z)\in\partial^0\phi(x^*(\cdot),u^*(\cdot),z^*).$$

Let

$$\Lambda=\left\{\lambda'\in R^l:\lambda'=(\lambda'_1,\ldots,\lambda'_l)\in R^l_+,\sum_{i=1}^{l}\lambda'_i\leqslant 1\right\}.$$

THEOREM 4.1. *Assume that* $f_i(x(\cdot, u(\cdot), z), u(\cdot), z)$, $i = 1, \dots, l$ *are subdifferentially regular at* $(u^*(\cdot), z^*)$. *If* $(u^*(\cdot), z^*)$ *is properly efficient to (MDOC), then there exists* $r^* > 0$ *such that*

$$\inf_{(\delta u(\cdot),\delta z)} \max_{\tilde g^{\lambda'}} \Bigg\{ \sum_{k=0}^{N-1} \left(\sum_{t=k+1}^{N} A^{t-k-1} B g^{\lambda'}_{x(t)} + g^{\lambda'}_{u(k)} \right)^T \delta u(\cdot) + \sum_{k=1}^{N} \Bigg(\Bigg[A^k \frac{\partial x^0(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots I)C \Bigg] g^{\lambda'}_{x(k)} + g^{\lambda'}_z \Bigg)^T \delta z : \ \lambda' \in \Lambda \Bigg\} \geqslant 0,$$

where

$$(\delta u(\cdot), \delta z) \in CI(u^*(\cdot), z^*) \bigcap CE(u^*(\cdot), z^*),$$

$$\tilde g^{\lambda'} = \sum_{i=1}^{l} (1 + \lambda'_i r^*) \tilde g^i,$$

$$\tilde g^i = (g^i_{x(1)}, \dots, g^i_{x(N)}, g^i_{u(0)}, \dots, g^i_{u(N-1)}, g^i_z) \in \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*).$$

Proof. Since $(u^*(\cdot), z^*)$ is a properly efficient solution of (MDOC), by Theorem 3.1, there exists $r^* > 0$ such that $(u^*(\cdot), z^*)$ is optimal to (DOC), where

$$\phi(x(\cdot), u(\cdot), z) = \sum_{i=1}^{l} f_i(x(\cdot), u(\cdot), z) + r^* \max\{f_1(x(\cdot), u(\cdot), z) - f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x(\cdot), u(\cdot), z) - f_l(x^*(\cdot), u^*(\cdot), z^*), 0\}.$$

Noticing that f_i $(i = 1, \dots, l)$ are subdifferentially regular at $(u^*(\cdot), z^*)$, by Proposition 2.3.12 in Clarke [4], $\max\{f_1(x(\cdot), u(\cdot), z) - f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x(\cdot), u(\cdot), z) - f_l(x^*(\cdot), u^*(\cdot), z^*), 0\}$ is subdifferentially regular at $(u^*(\cdot), z^*)$. It follows from Proposition 2.3.6 (c) in Clarke [4] that ϕ is subdifferentially regular at $(u^*(\cdot), z^*)$. Moreover,

$$\partial^0 \phi(x^*(\cdot), u^*(\cdot), z^*) \subseteq \sum_{i=1}^{l} \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*) + r^* \bigcup_{\lambda'_i \geqslant 0, \sum_{i=1}^{l} \lambda'_i \leqslant 1} \left[\sum_{i=1}^{l} \lambda'_i \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*) \right]$$
$$= \bigcup_{\lambda'_i \geqslant 0, \sum_{i=1}^{l} \lambda'_i \leqslant 1} \left[\sum_{i=1}^{l} (1 + \lambda'_i r^*) \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*) \right].$$

Let $\lambda' = (\lambda'_1, \dots, \lambda'_l)$. The results follow from Lemma 4.2. □

LEMMA 4.3. *Let $\Omega \subset R^q$ be a nonempty, compact and convex set and $h(x) = \max_{\omega\in\Omega} \omega^T x$. Then $\partial^0 h(0) = \Omega$.*

Proof. The is a direct consequence of Corollary 4.4.4 in Hiriart-Urruty and Lemmarechal [7]. □

The following theorem provides an alternative necessary condition for $(u*(\cdot), z^*)$ to be a properly efficient solution of (MDOC).

THEOREM 4.2. *Assume that f_i $(i = 1, \dots, l)$ are subdifferentially regular at $(u^*(\cdot), z^*)$. If $(u^*(\cdot), z^*)$ is a properly efficient solution of (MDOC). Then there exist $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*) \in intR_+^l$, $\mu_j \geqslant 0$, $j \in I(u^*(\cdot), z^*)$, $\nu_i \in R, i \in E$, $\tilde{g}^i = (g^i_{x(1)}, \dots, g^i_{x(N)}, g^i_{u(0)}, \dots, g^i_{u(N-1)}, g^i_z) \in \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*), i = 1, \dots, l$ such that*

$$\sum_{k=0}^{N-1}\left[\sum_{t=k+1}^{N} A^{t-k-1}B\left(\sum_{i=1}^{l}\lambda_i^* g^i_{x(t)}\right) + \sum_{i=1}^{l}\lambda_i^* g^i_{u(k)})\right] + \sum_{j\in I(u(\cdot),z)} \mu_j a_j + \sum_{i\in E}\nu_i a_i = 0,$$

$$\sum_{k=1}^{N}\left(\left[A^k\frac{\partial x^0(z)}{\partial z} + (A^{k-1} + A^{k-2} + \dots + I)C\right]\left(\sum_{i=1}^{l}\lambda_i^*\tilde{g}^i_{x(k)}\right) + \sum_{i=1}^{l}\lambda_i^*\tilde{g}^i_z\right) + \sum_{j\in I(u(\cdot),z)}\mu_j b_j + \sum_{i\in E}\nu_i b_i = 0.$$

Proof. By Theorem 4.1, $(\delta u(\cdot), \delta z) = (0, 0)$ solves the following scalar mathematical programming:

$$\begin{array}{lll}
(\mathrm{P}) & \min & F(\delta u(\cdot), \delta z) \\
 & \text{s.t.} & a_i^T \delta u(\cdot) + b_i^T \delta z = 0, i \in E, \\
 & & a_j^T \delta u(\cdot) + b_j^T \delta z \leqslant 0, j \in I(u(\cdot), z),
\end{array}$$

where

$$I(u(\cdot), z) = \{j \in I : a_i^T u^*(\cdot) + b_j^T z^* = c_j\},$$

$$F(\delta u(\cdot), \delta z) = \max_{v\in V'}\left\{v^T\begin{pmatrix}\delta u(\cdot)\\ \delta z\end{pmatrix} : v \in V'\right\},$$

$$V' = \left\{ \begin{pmatrix} \sum_{k=0}^{N-1} \left(\sum_{t=k+1}^{N} A^{t-k-1} B g_{x(t)}^{\lambda'} + g_{u(k)}^{\lambda'} \right) \\ \sum_{k=1}^{N} \left(\left[A^k \frac{\partial x^0(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots I)C \right] g_{x(k)}^{\lambda'} + g_z^{\lambda'} \right) \end{pmatrix} : \lambda' \in \Lambda \right\},$$

$$\tilde{g}^{\lambda'} = \sum_{i=1}^{l} (1 + \lambda_i' r^*) \tilde{g}^i,$$

$$\tilde{g}^i = (g_{x(1)}^i, \ldots, g_{x(N)}^i, g_{u(0)}^i, \ldots, g_{u(N-1)}^i, g_z^i) \in \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*).$$

It is easy to check that V' is a nonempty, compact and convex set. Note that (P) is a convex programming with only linear constraints. By Propositions 2.2.1 and 2.2.2 in Hiriart-Urruty and Lemmarechal [7], the necessary and sufficient condition for $(0, 0)$ to solve (P) is the existence of $\mu_j \geqslant 0$, $j \in I(\delta u(\cdot), \delta z)$ and $\nu_i \in R, i \in E$ such that

$$(0, 0) \in \partial^0 F(0, 0) + \sum_{j \in I(u(\cdot), z)} \mu_j \begin{pmatrix} a_j \\ b_j \end{pmatrix} + \sum_{i \in E} \nu_i \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

Applying Lemma 4.3, we get

$$(0, 0) \in V' + \sum_{j \in I(u(\cdot), z)} \mu_j \begin{pmatrix} a_j \\ b_j \end{pmatrix} + \sum_{i \in E} \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

It follows that there exist $\overline{\lambda} \in \Lambda$ and

$$\tilde{g}^{\overline{\lambda}'} = \sum_{i=1}^{l} (1 + \overline{\lambda}_i' r^*) \tilde{g}^i,$$

(where $\tilde{g}^i = (g_{x(1)}^i, \ldots, g_{x(N)}^i, g_{u(0)}^i, \ldots, g_{u(N-1)}^i, g_z^i) \in \partial^0 f_i(x^*(\cdot), u^*(\cdot), z^*)$, $i = 1, \ldots, l$) such that

$$\sum_{k=0}^{N-1} \left[\sum_{t=k+1}^{N} A^{t-k-1} B \left(\sum_{i=1}^{l} (1 + \overline{\lambda}_i' r^*) g_{x(t)}^i \right) + \sum_{i=1}^{l} (1 + \overline{\lambda}_i' r^*) g_{u(k)}^i \right] + \sum_{j \in I(u(\cdot), z)} \mu_j a_j + \sum_{i \in E} \nu_i a_i = 0,$$

$$\sum_{k=1}^{N} \left(\left[A^k \frac{\partial x^0(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots + I)C \right] \left(\sum_{i=1}^{l} (1 + \overline{\lambda}_i' r^*) \tilde{g}_{x(k)}^i \right) + \sum_{i=1}^{l} (1 + \overline{\lambda}_i' r^*) \tilde{g}_z^i \right) + \sum_{j \in I(u(\cdot), z)} \mu_j b_j + \sum_{i \in E} \nu_i b_i = 0.$$

Set $\lambda_i^* = (1+\overline{\lambda}_i' r^*)$, $i = 1, \dots, l$, $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*)$. It is obvious that $\lambda^* \in \text{int} R_+^l$. The conclusion holds. □

REMARK 4.2. It is easy to see that Theorems 4.1 and 4.2 hold even if $(u^*(\cdot), z^*)$ is a locally properly efficient solution of (MDOC).

In what follows, we shall discuss how to apply Theorem 4.1 or Theorem 4.2 to check whether a control-parameter pair $(u^*(\cdot), z^*)$ is optimal to a discrete time optimal control problem with a scalar composite performance measure under some special conditions.

Consider the following discrete time optimal control problem with a scalar composite performance measure:

$$\text{(CDOC)} \quad \min \quad h(f_1(x(\cdot), u(\cdot), z), \dots, f_l(x(\cdot), u(\cdot), z))$$
$$\text{s.t.} \quad (u(\cdot), z) \in \mathcal{F}.$$

Yang and Teo [19] discussed necessary optimality conditions for (CMOC) with $l = 2$. They first showed that (CMOC) (with $l = 2$) admits at least one optimal control-parameter pair $(u^*(\cdot), z^*)$ which is also an efficient solution of (MDOC) (with $l = 2$). Under some weaker condition than convexity, they showed $(u^*(\cdot), z^*)$ is also an optimal control-parameter pair to a scalar optimal control problem whose objective function is a convex composition of the two original objective functions. Finally, they derived a necessary condition (Yang and Teo [19], Theorem 3.3) for $(u^*(\cdot), z^*)$ to be optimal for the scalar optimal control problem whose objective function is a convex composition of the two original objective functions.

Following the proof of Lemma 1 in Geoffrion [6], it can be shown that if $h(y_1, \dots, y_l)$ is increasing with respect to each of its component, and $\mathcal{F}$ is non-empty and compact or $\max\{f_1(x(\cdot), u(\cdot), z), \dots, f_l(x(\cdot), u(\cdot), z)\} \to +\infty$ as $\|(x(\cdot), u(\cdot), z)\| \to +\infty$, then (CDOC) admits at least one optimal control-parameter pair $(u^*(\cdot), z^*)$ which is also efficient to (MDOC).

In the following we provide a sufficient condition for a local proper efficient solution. We assume that

(A_1) $h(y_1, \dots, y_l)$ is continuously differentiable at
$y^* = (y_1^*, \dots, y_l^*) = (f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x^*(\cdot), u^*(\cdot), z^*))$
and $\frac{\partial h}{\partial y_i}|_{y=y^*} > 0$, $i = 1, \dots, l$.

PROPOSITION 4.1. *Assume that* (A_1) *holds and that* $f_i(x(\cdot), u(\cdot), z)$ $(i = 1, \dots, l)$ *are continuous at* $(u^*(\cdot), z^*)$. *If* $(u^*(\cdot), z^*)$ *is an optimal solution of (CDOC) and an efficient solution of (MDOC), then* $(u^*(\cdot), z^*)$ *is a locally properly efficient solution of (MDOC).*

Proof. Since $(u^*(\cdot), z^*)$ is an efficient solution of (MDOC), we only need to show that it is also locally proper efficient. Suppose to the contrary that there exists a sequence $\{(u^k(\cdot), z^k)\}$ with $(u^k(\cdot), z^k) \to (u^*(\cdot), z^*)$ and $i^* \in \{1, \dots, l\}$ such that

$$\frac{f_i(x^*(\cdot), u^*(\cdot), z^*) - f_i(x^k(\cdot), u^k(\cdot), z^k)}{f_j(x^k(\cdot), u^k(\cdot), z^k) - f_j(x^*(\cdot), u^*(\cdot), z^*)} \to +\infty \tag{21}$$

as $k \to +\infty$.

Since $(u^*(\cdot), z^*)$ solves (CDOC), we have

$$\begin{aligned} & h(f_1(x^k(\cdot), u^k(\cdot), z^k), \dots, f_l(x^k(\cdot), u^k(\cdot), z^k)) \\ & \quad - h(f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x^*(\cdot), u^*(\cdot), z^*)) \\ & = \sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \frac{\partial h}{\partial y_i}|_{y=\xi^k} [f_i(x^k(\cdot), u^k(\cdot), z^k) - f_i(x^*(\cdot), u^*(\cdot), z^*)] \\ & \quad + \frac{\partial h}{\partial y_{i^*}}|_{y=\xi^k} [f_{i^*}(x^k(\cdot), u^k(\cdot), z^k) - f_{i^*}(x^*(\cdot), u^*(\cdot), z^*)] \geqslant 0, \end{aligned} \tag{22}$$

where k is large enough and ξ^k lies in the segment

$$\begin{aligned} & [(f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x^*(\cdot), u^*(\cdot), z^*)), \\ & \quad (f_1(x^k(\cdot), u^k(\cdot), z^k), \dots, f_l(x^k(\cdot), u^k(\cdot), z^k))]. \end{aligned}$$

By the assumption of the proposition, we see that

$$\xi^k \to y^* = (f_1(x^*(\cdot), u^*(\cdot), z^*), \dots, f_l(x^*(\cdot), u^*(\cdot), z^*))$$

as $k \to +\infty$, and without loss of generality, we assume that

$$\frac{\partial h}{\partial y_{i^*}}|_{y=\xi^k} > 0, i = 1, \dots, l.$$

Consequently, (22) implies

$$\begin{aligned} & \left(\sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \frac{\partial h}{\partial y_i}|_{y=\xi^k} \right) \max_{i \in \{1,\dots,l\} \setminus \{i^*\}} \{f_i(x^k(\cdot), u^k(\cdot), z^k) - f_i(x^*(\cdot), u^*(\cdot), z^*)\} \\ & \geqslant \frac{\partial h}{\partial y_{i^*}}|_{y=\xi^k} [f_{i^*}(x^*(\cdot), u^*(\cdot), z^*) - f_{i^*}(x^k(\cdot), u^k(\cdot), z^k)]. \end{aligned}$$

That is,

$$\begin{aligned} & \frac{f_{i^*}(x^*(\cdot), u^*(\cdot), z^*) - f_{i^*}(x^k(\cdot), u^k(\cdot), z^k)}{\max_{i \in \{1,\dots,l\} \setminus \{i^*\}} \{f_i(x^k(\cdot), u^k(\cdot), z^k) - f_i(x^*(\cdot), u^*(\cdot), z^*)\}} \\ & \quad \leqslant \frac{\sum_{i \in \{1,\dots,l\} \setminus \{i^*\}} \frac{\partial h}{\partial y_i}|_{y=\xi^k}}{\frac{\partial h}{\partial y_{i^*}}|_{y=\xi^k}}. \end{aligned} \tag{23}$$

Inequality (23) contradicts (21) as $k \to +\infty$. The proof is complete. □

In view of Remark 4.2 and Proposition 4.1, if $(u^*(\cdot), z^*)$ is both an optimal solution of (CDOC) and an efficient solution of (MDOC) (and if the conditions of Proposition 4.1 hold), in addition, f_i $(i = 1, \dots, l)$ are subdifferentially regular at $(u^*(\cdot), z^*)$, then Theorems 4.1 or 4.2 serve as necessary optimality conditions.

Now we take a look at the examples of h cited in Geoffrion [6] (where 'maximize' and 'min' should here be replaced by 'minimize' and 'max', respectively). For models B, C, D, E in Geoffrion [6], any solution $x^* \in X$ to the composite problem is an efficient solution of the bicriteria optimization problem, and the condition $\partial h/\partial y_i|_{y=y^*} > 0, i = 1, 2$ also holds, where $y^* = (f_1(x^*), f_2(x^*))$, thus x^* is a locally properly efficient solution of the bicriteria optimization problem, by Proposition 4.1.

Model A should here be rewritten as

$$\min_{x\in X} \max\{f_1(x), f_2(x)\}.$$

For this model, we cite a simple example to illustrate the fact that there may exist no optimal solution which is also locally proper efficient to the corresponding bicriteria optimization problem:

$$\min\{(f_1(x), f_2(x)) : x \in X\}.$$

Let $X = [0, +\infty) \subset R$, $f_1(x) = -x$, $f_2(x) = x^2, \forall x \in X$. Then $x^* = 0$ is optimal to the problem:$\min_{x\in X} \max\{f_1(x), f_2(x)\}$ and $x^* = 0$ is also efficient to the corresponding bicriteria optimization problem. However, it is straightforward to verify that $x^* = 0$ is not a locally properly efficient solution of the corresponding bicriteria optimization problem. Despite this fact, we can use the following approximate model for model A:

$$(A_k) \qquad \min_{x\in X} h_k(f_1(x), f_2(x)),$$

where

$$h_k(f_1(x), f_2(x)) = \frac{1}{k}\ln[\exp(kf_1(x)) + \exp(kf_2(x))].$$

If X is nonempty and compact, then it is easy to verify that

$$0 \leqslant h_k(f_1(x), f_2(x)) - h(f_1(x), f_2(x)) \leqslant \frac{\ln 2}{k} \max_{x\in X} \max\{f_1(x), f_2(x)\}.$$

Suppose that f_1 and f_2 are continuous on X. Then $\{h_k\}$ uniformly converges to $h(f_1(x), f_2(x))$ on X as $k \to +\infty$. Thus, to solve model A, we can approximately solve model (A_k). It is easy to see that any solution of model (A_k) is an efficient solution of the corresponding bicriteria optimization problem $\min\{(f_1(x), f_2(x)) :$

$x \in X\}$ and for model (A_k), the condition $\partial h_k/\partial y_i|_{y=y^*} > 0, i = 1, 2$ always holds. By Proposition 4.1 and Remark 4.2, the results of Theorems 4.1 and 4.2 are applicable to (A_k).

5. Conclusions

We presented new characterizations of properly efficient solutions of a nonconvex multiobjective optimization problem in terms of the stability of one scalar constrained optimization problem and the existence of an exact penalty function for a scalar constrained optimization problem, respectively. Applying one of the characterizations, we derived necessary conditions for a properly efficient control-parameter pair of a multicriteria discrete time optimal control problem with linear state equations. Finally, we analyzed the applicability of the necessary conditions to discrete optimal control problems with a composite performance measure.

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References

1. Benson, H. P. and Morin, T. L. (1977), The vector maximization problem: proper efficiency and stability, *SIAM Journal on Applied Mathematics* 32, 64–72.
2. Borwein, J. M. (1977), Proper efficient points for maximization with respect to cones, *SIAM Journal on Control and Optimization* 15, 57–63.
3. Choo, E. U. and Atkins, D. R. (1983), Proper Efficiency in Nonconvex Multicriteria Programming, *Mathematics of Operations Research* 8, 467–470.
4. Clarke, F. H. (1983), *Optimization and Nonsmooth Analysis,* John Wiley & Sons, New York.
5. Deng, S. (1998), on efficient solutions in vector optimization, *Journal of Optimization Theory and Applications* 96, 201–209.
6. Geffrion, A. M. (1967), Solving bicritrion mathematical programs, *Operations Research* 15, 39–45.
7. Hiriart-Urruty J. B. and Lemmarechal, C. (1993), *Convex Analysis and Minimization Algorithms I,* Springer, Berlin.
8. Jahn, J. (1985), A characterization of properly minimal elements of a set, *SIAM Journal on Control and Optimization* 23, 649–656.
9. Jennings, L. S., Fischer, M. E., Teo, K. L. and Goh, C. J. (1990), MISER3, Optimal Control Software, Theory and User Manual, EMCOSS Pty Ltd, 7 Topaz Place, Carine, WA 6020, Australia.
10. Li, D. (1990), On the minimax solution of multiple linear quadratic problems, *IEEE Transactions on Automatic Control* 35, 1153–1156.
11. Li, D. (1993), On general multiple linear quadratic control problems, *IEEE Transactions on Automatic Control* 38, 1722–1726.
12. Liao, L. and Li, D. (2000), Successive method for general multiple linear quadratic control problems in discrete time, *IEEE Transactions on Automatic Control* 45, 1380–1384.
13. Liu, L. P. (1993), Characterization of nondominated controls in terms of solutions of weighting problems, *Journal of Optimization Theory and Applications* 77, 545–561.

14. Rosenberg, E. (1984), Exact penalty functions and stability in locally Lipschitz programming, *Mathematical Programming* 30, 340–356.
15. Salukvadze, M. (1974), On the existence of solutions in problems of optimization under vector-valued criteria, *Journal of Optimization Theory and Applications* 13, 203–217.
16. Sawaragi, Y., Nakayama, H. and Tanino, T. (1985), *Theory of Multiobjective Optimization*, Academic Press, New York.
17. Teo, K. L., Goh, C. J. and Wong, K. H. (1990), *A Unified Computational Approach to Optimal Control Problems*, Logman Scientific and Technical, New York.
18. Toivonen, H. T. (1984), A multiobjective linear quadratic Gaussian control problem, *IEEE Transactions on Automatic Control* 29, 279–280.
19. Yang, X. Q. and Teo, K. L. (1999), Necessary optimality conditions for bicriterion discrete time optimal control problems, *Journal of the Australian Mathematical Society, Ser. B* 40, 392–402.
20. Yu, P. L. and Leitmann, G. (1974), Nondominated decision and cone convexity in dynamic multicriteria decision problems, *Journal of Optimization Theory and Applications* 14, 573–584.
21. Zadeh, L. A. (1963), Optimality and non-scalar-valued performance criteria, *IEEE Transactions on Automatic Control 8*, 59–60.
22. Zalmai, G. L. (1996), Proper efficiency and duality for a class of constrained multiobjective fractional optimal control problems containing arbitrary norms, *Journal of Optimization Theory and Applications* 90, 435–456.